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Excitation of waves trapped by submerged slender structures, and nonlinear resonance

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In a companion paper the existence of trapped waves over submerged cylinders has been analysed, and a necessary condition for their excitation was derived. In the present paper, this study is extended to obtain physically more important results. First we consider a more realistic geometry, namely a finite, although slender, cylinder. Second we derive the necessary and sufficient conditions for the excitation of trapped modes; and lastly, the induced resonant response is studied with the multiple-scales technique. It is shown then that the wave amplitude satisfies an equation similar to the resonant nonlinear oscillator.

1. Introduction

In an earlier paper (Aranha 1988, hereinafter referred to as I) the existence has been demonstrated of trapped modes over a submerged cylinder for an otherwise arbitrary body geometry and wave frequency. These modes are non-trivial solutions of the potential equation and homogeneous free-surface, bottom and body boundary conditions. Furthermore they have the form (see (2.5) in **I**; x = cylinder axis)

$$\left.\begin{aligned} \hat{T}(x,y,z,t) &= T(y,z)\,\mathrm{e}^{\mathrm{i}(K_\mathrm{T}x-\Omega t)}; \quad K_\mathrm{T} = K_\mathrm{T}(\Omega),\\ T(y,z) &\sim t^{\pm}\,\mathrm{e}^{-\lambda_0|y|}\,\mathrm{e}^{K_0 z}, \quad y \rightarrow \pm\infty,\\ \frac{\lambda_0}{K_0} &= \left(\left(\frac{K_\mathrm{T}}{K_0}\right)^2-1\right)^{\frac{1}{2}}; \quad K_0 = \Omega^2. \end{aligned}\right\} \tag{1.1}$$

In the above expression and throughout this work, the deep-water condition has been assumed, and all variables have been non-dimensionalized by the cross-section beam B and acceleration due to gravity g. As shown in I, the trapped mode $\{K_\mathrm{T}; T(y,z)\}$ can be determined by a standard eigenvalue problem, but since $K_\mathrm{T} > K_0$, only nonlinear interactions between two harmonic waves can excite it. If $\omega_j, j = 1,2$, are the frequencies of these waves and $\alpha_j, j = 1,2$, the angles they make with the longitudinal axis then a necessary condition for trapped-mode excitation is given by the relations

$$\left.\begin{aligned} \Omega &= \omega_2-\omega_1,\\ K_\mathrm{T}(\Omega) &= \omega_2^2\cos\alpha_2-\omega_1^2\cos\alpha_1, \end{aligned}\right\} \tag{1.2}$$

since only then will the nonlinear interaction of the incoming waves have a longitudinal factor that can be matched with $\exp \mathrm{i}(K_\mathrm{T}x-\Omega t)$ (see (1.1)).

In the present work, results derived in I will be extended in several ways. First, a more realistic geometry – a finite, although slender, cylinder with length $L = 1/\epsilon$, $\epsilon \ll 1$ – will be considered here. Also, necessary and sufficient conditions for trapped-mode excitation will be established, and the induced resonant response will be analysed by multiple scales, a standard technique for nonlinear oscillations (see Kevorkian & Cole 1985). As it might be expected, a cubic nonlinear wave equation is obtained, but some particular features deserve special attention. First, the nonlinear wave equation is non-dispersive, and so soliton-like behaviour should not be expected here. Second, the trapped mode leaks (to higher order) some energy to infinity, giving rise to a leading-order nonlinear radiation damping. Also, the expression for the detuning parameter has some peculiarities in itself, that have both practical and theoretical consequences. Finally, the body slenderness introduces a second slow lengthscale, $X_1 = \epsilon x$, beside the one naturally introduced by the nonlinearity. It is certainly of some interest to know how this second lengthscale interferes with the whole problem.

Once the trapped mode $\{K_T; T(y,z)\}$ at the frequency Ω is determined, the following coefficients can be found:

$$\left.\begin{aligned} I_0(\Omega) &= \Omega\int_{-\infty}^{\infty} T^2(y,0)\,\mathrm{d}y; & I_1(\Omega) &= K_T\iint_{A_\infty} T^2(y,z)\,\mathrm{d}A_\infty;\\ I_2(\Omega) &= \int_{-\infty}^{\infty} T^4(y,0)\,\mathrm{d}y; & I_3(\Omega) &= \int_{-\infty}^{\infty} T^2(y,0)\left(\frac{\partial T}{\partial y}(y,0)\right)^2\mathrm{d}y;\\ I_4(\Omega) &= \int_{-\infty}^{\infty}\left(\frac{\partial T}{\partial y}(y,0)\right)^4\mathrm{d}y; & I_5(\Omega) &= \int_{\partial B}\left[\left(\frac{\partial T}{\partial s}\right)^2 + K_T^2 T^2(s)\right]\mathrm{d}\,\partial B. \end{aligned}\right\} \tag{1.3}$$

In (1.3), A_∞ is the entire fluid region in the cross-section plane (y,z) and ∂B is the contour line of the submerged cylinder. As it will become clear in this work, the coefficients (1.3) will be needed to determine the nonlinear response. In particular, if $c(\Omega)$ is the trapped wave 'group velocity' in the longitudinal x-direction, then, from (4.2) in I and (1.3) above, it follows that

$$c(\Omega) = \frac{\mathrm{d}\Omega}{\mathrm{d}K_T} = \frac{I_1(\Omega)}{I_0(\Omega)}. \tag{1.4}$$

2. Trapped-mode excitation

In this section the excitation of trapped modes, due to the non-linear interaction between two waves with amplitudes A_j, frequencies ω_j and angles of incidence α_j, $j = 1, 2$, will be studied. Some preliminary definitions will be introduced first. Thus, let

$$A_0 = (A_1 A_2)^{\frac{1}{2}}, \quad a_j = \frac{A_j}{A_0}, \quad j = 1,2, \quad \delta = \frac{A_0}{B}, \tag{2.1}$$

where δ is the small wave-amplitude parameter. Since B is the lengthscale and $(Bg)^{\frac{1}{2}}$ the related velocity scale, the non-dimensional potential Φ is defined by

$$\Phi = \frac{1}{B(Bg)^{\frac{1}{2}}}\tilde{\Phi} \sim O(\delta), \tag{2.2}$$

where a tilde denotes a dimensional variable. If an asterisk stands for the complex conjugate, then, from (2.1), (2.2), the incident wave is given by

$$\left.\begin{aligned} \Phi_{I,j}(x,y,z,t) &= \delta\tfrac{1}{2}a_j\,\phi_{I,j}(y,z;\alpha_j)\,\mathrm{e}^{\mathrm{i}(\omega_j^2 x\cos\alpha_j - \omega_j t)} + (*),\\ \phi_{I,j}(y,z;\alpha_j) &= \frac{1}{\omega_j}\,\mathrm{e}^{\omega_j^2 z}\,\mathrm{e}^{\mathrm{i}\omega_j^2 y\sin\alpha_j}. \end{aligned}\right\} \tag{2.3}$$

The linear diffraction potential at the frequency ω_j can then be written as

$$\left.\begin{aligned}\Phi_j(x,y,z,t) &= \phi_j(y,z;\epsilon x,\alpha_j)\,\mathrm{e}^{\mathrm{i}(\omega_j^2 x\cos\alpha_j-\omega_j t)}+(*),\\ \phi_j(y,z;\epsilon x;\alpha_j) &= \phi_{\mathrm{I},j}(y,z;\alpha_j)+\phi_{\mathrm{s},j}(y,z;\epsilon x;\alpha_j),\end{aligned}\right\}\tag{2.4}$$

where $\phi_{\mathrm{s},j}(y,z;\epsilon x;\alpha_j)$ is the related scattered wave. Since the fast-changing factor $\exp(\mathrm{i}\,\omega_j^2\,x\cos\alpha_j)$ has been factored out, the scattered wave $\phi_{\mathrm{s},j}$ changes slowly with x for a slender body. Or, more precisely, its longitudinal derivative is of relative order ϵ where ϵ is the small slenderness parameter. This fact is recognized here by writing ϕ_j and $\phi_{\mathrm{s},j}$ as functions of the slow variable ϵx (see (2.4)).

If the two harmonic waves are acting together the linear response is given by $\delta\cdot\Phi_{\mathrm{L}}(x,y,z,t)$, where

$$\Phi_{\mathrm{L}}(x,y,z,t) = \sum_{j=1}^{2}\left[\tfrac{1}{2}a_j\,\phi_j(y,z;\epsilon x;\alpha_j)\,\mathrm{e}^{\mathrm{i}(\omega_j^2 x\cos\alpha_j-\omega_j t)}+(*)\right].\tag{2.5}$$

$\delta\cdot\Phi_{\mathrm{L}}(x,y,z,t)$ is the leading-order solution of the nonlinear problem

$$\nabla^2\Phi+\frac{\partial^2\Phi}{\partial x^2}=0;\quad \boldsymbol{\nabla}=\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z},\tag{2.6a}$$

$$\left.\frac{\partial^2\Phi}{\partial t^2}+\frac{\partial\Phi}{\partial z}\right|_{z=0} = Q[\Phi;\Phi]+C[\Phi;\Phi;\Phi]+\dots,\tag{2.6b}$$

$$\boldsymbol{\nabla}\Phi\cdot\boldsymbol{n}|_{\partial B}=0,\tag{2.6c}$$

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=-h}=0,\tag{2.6d}$$

$$\text{Radiation condition.}\tag{2.6e}$$

The symbol ... stands for higher-order terms, and later it will be used also to indicate terms that, although not of a higher order, are unimportant for trapped-mode excitation. Owing to the nonlinear character of the problem, the proper radiation condition will be discussed in the next section. At order δ, however, it must be given by the incident wave (2.4) and the classical linear radiation condition for the scattered potentials $\phi_{\mathrm{s},j}$. To simplify the exposition the body will be assumed fixed in space and so the relevant boundary condition is (2.6c). In this circumstance the free surface is the only source of non-linearities, represented here by the quadratic and cubic operators shown in (2.6b). If

$$\hat{\boldsymbol{\nabla}}=\boldsymbol{i}\frac{\partial}{\partial x}+\boldsymbol{j}\frac{\partial}{\partial y}+\boldsymbol{k}\frac{\partial}{\partial z},\tag{2.7}$$

these operators can be written as

$$\left.\begin{aligned}Q[\Phi;\Psi] &= \left[-2\hat{\boldsymbol{\nabla}}\Phi\hat{\boldsymbol{\nabla}}\left(\frac{\partial\Psi}{\partial t}\right)+\frac{\partial\Phi}{\partial t}\frac{\partial}{\partial z}\left(\frac{\partial^2\Psi}{\partial t^2}+\frac{\partial\Psi}{\partial z}\right)\right]_{z=0},\\ C[\Phi;\Psi;\Lambda] &= \left[-\tfrac{1}{2}\hat{\boldsymbol{\nabla}}\Phi\cdot\hat{\boldsymbol{\nabla}}(\hat{\boldsymbol{\nabla}}\Psi\cdot\hat{\boldsymbol{\nabla}}\Lambda)+\frac{\partial\Phi}{\partial t}\frac{\partial}{\partial z}\left(2\hat{\boldsymbol{\nabla}}\Psi\cdot\hat{\boldsymbol{\nabla}}\left(\frac{\partial\Lambda}{\partial z}\right)\right)\right.\\ &\quad+\left(-\frac{\partial\Phi}{\partial t}\frac{\partial^2\Psi}{\partial z\,\partial t}+\tfrac{1}{2}\hat{\boldsymbol{\nabla}}\Phi\cdot\hat{\boldsymbol{\nabla}}\Psi\right)\frac{\partial}{\partial z}\left(\frac{\partial^2\Lambda}{\partial t^2}+\frac{\partial\Lambda}{\partial z}\right)\\ &\quad\left.-\frac{1}{2}\frac{\partial\Phi}{\partial t}\frac{\partial\Psi}{\partial t}\frac{\partial^2}{\partial z^2}\left(\frac{\partial^2\Lambda}{\partial t^2}+\frac{\partial\Lambda}{\partial z}\right)\right]_{z=0};\end{aligned}\right\}\tag{2.8}$$

see Newman (1978) for details. If $\{\omega_j;\alpha_j; j=1,2\}$ satisfy (1.2) it is an easy task to check that

$$Q[\delta\cdot\Phi_{\mathrm{L}};\delta\cdot\Phi_{\mathrm{L}}] = \delta^2[q(y;\epsilon x)\,\mathrm{e}^{\mathrm{i}(K_{\mathrm{T}}x-\Omega t)}+(*)]+\ldots, \tag{2.9}$$

where ... now indicates unimportant terms, and

$$q(y;\epsilon x) = -\tfrac{1}{4}\mathrm{i}\left[\Gamma\phi_1^*\phi_2+\Omega\frac{\partial\phi_1^*}{\partial y}\frac{\partial\phi_2}{\partial y}+\omega_2\frac{\partial}{\partial y}\left(\phi_2\frac{\partial\phi_1^*}{\partial y}\right)-\omega_1\frac{\partial}{\partial y}\left(\phi_1^*\frac{\partial\phi_2}{\partial y}\right)\right]_{z=0}, \tag{2.10a}$$

with

$$\Gamma = 2\omega_1^2\omega_2^2\Omega(1-\cos\alpha_1\cos\alpha_2)-\omega_2^4\omega_1\sin^2\alpha_2+\omega_2\omega_1^4\sin^2\alpha_1. \tag{2.10b}$$

In deriving (2.10), the potentials ϕ_j were considered independent of x since, owing to body slenderness, $\partial\phi_j/\partial x\sim O(\epsilon)$ and terms of order $\epsilon\delta^2$ were neglected when compared with δ^2. The second-order potential Φ_2 can be written as

$$\Phi_2(x,y,z,t) = \delta^2[\phi_{21}(x,y,z,t)\,\mathrm{e}^{\mathrm{i}(K_{\mathrm{T}}x-\Omega t)}+(*)]+\ldots, \tag{2.11}$$

where $\phi_{21}(x,y,z,t)$ is the solution of the problem (see (2.6))

$$\nabla^2\phi_{21}-K_{\mathrm{T}}^2\phi_{21}+\left[2\mathrm{i}K_{\mathrm{T}}\frac{\partial\phi_{21}}{\partial x}+\frac{\partial^2\phi_{21}}{\partial x^2}\right]=0, \tag{2.12a}$$

$$\frac{\partial\phi_{21}}{\partial z}-\Omega^2\phi_{21}\big|_{z=0} = \left[2\mathrm{i}\Omega\frac{\partial\phi_{21}}{\partial t}-\frac{\partial^2\phi_{21}}{\partial t^2}\right]_{z=0}+q(y;\epsilon x), \tag{2.12b}$$

$$\nabla\phi_{21}\cdot\boldsymbol{n}\big|_{\partial B}=0, \tag{2.12c}$$

$$\left.\frac{\partial\phi_{21}}{\partial z}\right|_{z=-h}=0, \tag{2.12d}$$

subject also to boundary conditions at infinity and initial conditions. If A_∞ is the fluid region in the plane (y,z) and $\Psi(y,z)$ an arbitrary exponentially decaying function as $|y|\to\infty$, then multiplying (2.12a) by $\Psi(y,z)$ and integrating by parts in A_∞, one obtains

$$M_\infty(\phi_{21};\Psi)-\left[\iint_{A_\infty}\left(2\mathrm{i}K_{\mathrm{T}}\frac{\partial\phi_{21}}{\partial x}+\frac{\partial^2\phi_{21}}{\partial x^2}\right)\Psi\,\mathrm{d}A_\infty+\int_{-\infty}^{\infty}\left(2\mathrm{i}\Omega\frac{\partial\phi_{21}}{\partial t}(y,0)\right.\right.$$
$$\left.\left.-\frac{\partial^2\phi_{21}}{\partial t^2}(y,0)\right)\Psi(y,0)\,\mathrm{d}y\right]=\int_{-\infty}^{\infty}q(y;\epsilon x)\,\Psi(y,0)\,\mathrm{d}y, \tag{2.13}$$

where

$$M_\infty(\phi_{21};\Psi) = \iint_{A_\infty}[\nabla\phi_{21}\cdot\nabla\Psi+K_{\mathrm{T}}^2\phi_{21}\Psi]\,\mathrm{d}A_\infty-\Omega^2\int_{-\infty}^{\infty}\phi_{21}(y,0)\,\Psi(y,0)\,\mathrm{d}y. \tag{2.14}$$

Since $q(y;\epsilon x)$ changes slowly with x then, with an error $O(\epsilon)$, one may try a solution ϕ_{21} of (2.13) independent of x and t. In this case the terms within square brackets in (2.13) are zero, and taking $\Psi=T$ one obtains

$$M_\infty(\phi_{21};T)=I_0(\Omega)\,Q_{\mathrm{L}}(\epsilon x), \tag{2.15}$$

where $I_0(\Omega)$ is defined in (1.3) and $Q_L(\epsilon x)$ is given by

$$Q_L(\epsilon x) = \frac{1}{I_0(\Omega)} \int_{-\infty}^{\infty} q(y;\, \epsilon x)\, T(y,0)\, \mathrm{d}y. \tag{2.16}$$

Since $\{K_T; T(y,z)\}$ is a trapped mode at frequency Ω, then $M_\infty(T;\phi) = 0$ for an arbitrary ϕ; see I for details. Then either $Q_L \equiv 0$, in which case the excitation is orthogonal to the trapped mode, or (2.15) leads to a contradiction, synonymous with resonance. The necessary and sufficient conditions for trapped-mode excitation are then: (i) condition (1.2) is satisfied; (ii) $Q_L \not\equiv 0$. Since this latter condition is usually satisfied, resonance is expected if the tuning conditions (1.2) are fulfilled.

The ensuing nonlinear resonant response will be analysed in the next section, but some insight can be gained by looking at the linear solution of (2.12). To avoid the contradictory result (2.15), one should look now for a particular solution ϕ_{21}, a function of x and t. In this way let

$$\phi_{21}(x,y,z,t) = \tfrac{1}{4}\mathrm{i}\, Q_L(\epsilon x)\left(\frac{x}{c(\Omega)} + t\right) T(y,z) + \Delta\phi_{21}(y,z), \tag{2.17}$$

with $c(\Omega)$ defined in (1.4). Placing (2.17) into (2.13) one obtains the identity

$$M_\infty(\Delta\phi_{21};\, \Psi) = -\tfrac{1}{2}Q_L(\epsilon x)\left[\frac{K_T}{c(\Omega)} \iint_{A_\infty} T\Psi\, \mathrm{d}A_\infty + \Omega \int_{-\infty}^{\infty} T(y,0)\, \Psi(y,0)\, \mathrm{d}y\right] + \int_{-\infty}^{\infty} q(y;\, \epsilon x)\, \Psi(y,0)\, \mathrm{d}y. \tag{2.18}$$

If $\Psi(y,z)$ is taken equal to $T(y,z)$, the left-hand side of (2.18) is zero again but the right-hand side is also now zero (see (1.3), (1.4) and (2.16)). So a particular solution of (2.12) exists of the form (2.17), and the complete solution can be obtained by adding a homogeneous one and imposing the relevant boundary and initial conditions.†

For large $(x;t)$ the solution (2.17) is dominated by the first contribution

$$\left.\begin{aligned} \phi_{21}(x,y,z,t) &\sim \tfrac{1}{2}A(x,t)\, T(y,z), \\ A(x,t) &= \tfrac{1}{2}\mathrm{i}Q_L(\epsilon x)\left(\frac{x}{c(\Omega)} + t\right). \end{aligned}\right\} \tag{2.19}$$

The above behaviour is typical for a linear resonant phenomenon. For a harmonic oscillator the linear resonant response increases linearly in time, whereas in the present case it increases linearly with time and longitudinal coordinate.

Some important conclusions can be derived from (2.19). First, (2.11) and (2.19) indicate that the potential Φ is a function of the slow time and length $\delta^2 t, \delta^2 x$, respectively, where δ is the small wave-amplitude parameter. At resonance the actual order of magnitude of the response should be larger than δ – of order $\delta^\beta, \beta < 1$, for example – and so Φ should be a function of the slow length and time scales $(X;T) = (\delta^\beta)^2\, (x;t)$. For $(X;T) \sim O(1)$ the resonant response is of order δ^β and it is dominated by the term $\delta^2 A(x,t)\, T(y,z)$, of order $\delta^2(x;t) = \delta^{2-2\beta}(X;T)$; see (2.11),

† Recall that (2.17) is correct with an error $O(\epsilon)$ since $Q_L(\epsilon x)$ changes slowly with x. In reality, if the body axis is on the segment $0 \leqslant x \leqslant 1/\epsilon$, then $Q_L(\epsilon x) \equiv 0$ for $x < 0$ and $x > 1/\epsilon$ and so (2.17) is bounded in the limits $x \to \pm\infty$.

(2.19). Since $(X;T) \sim O(1)$, the equality $\delta^{\beta} = \delta^{2-2\beta}$ must hold, leading to $\beta = \frac{2}{3}$. In this case,

$$X = \delta^{\frac{4}{3}}x, \qquad T = \delta^{\frac{4}{3}}t, \tag{2.20a}$$

and, to leading order, the potential is given by

$$\Phi(x,y,z,t) \sim \delta^{\frac{2}{3}}\tfrac{1}{2}A(X,T)\,T(y,z)\,\mathrm{e}^{\mathrm{i}(K_{\mathrm{T}}x-\Omega t)} + (*). \tag{2.20b}$$

The amplitude function $A(X,T)$ should be determined to avoid a secular term (2.19) at order δ^2 and, from (2.20b), a cubic nonlinear wave equation for $A(X,T)$ should be obtained. The arguments presented here are standard in nonlinear oscillation theory; see Kevorkian & Cole (1985) for details, and Aranha, Yue & Mei (1982) in the context of free-surface problems. In the next section the wave equation for $A(X,T)$ will be derived.

3. Nonlinear resonant response

In §§1 and 2 of the present work, the incoming incident waves were supposed to tune exactly to the trapped mode $\{K_{\mathrm{T}}(\Omega); T(y,z;\Omega)\}$; see (1.2). In reality a near-resonant condition must be considered and so, instead of (1.2), the following relations will be assumed hereafter:

$$\bar{\Omega} = \omega_2 - \omega_1, \quad \bar{K}_{\mathrm{T}} = \omega_2^2\cos\alpha_2 - \omega_1^2\cos\alpha_1, \tag{3.1a}$$

with $\{\bar{\Omega}; \bar{K}_{\mathrm{T}}\}$ close, in some sense, to $\{\Omega; K_{\mathrm{T}}\}$. Consistent with (2.20$a$), one may write

$$\bar{\Omega} = \Omega + \delta^{\frac{4}{3}}\Delta\Omega, \quad \bar{K}_{\mathrm{T}} = K_{\mathrm{T}} + \delta^{\frac{4}{3}}\Delta K_{\mathrm{T}}, \tag{3.1b}$$

where $(\Delta\Omega; \Delta K_{\mathrm{T}}) \sim O(1)$ defines the near-resonant region. This means that the 'width' of the resonant peak is of order $\delta^{\frac{4}{3}}$ although the actual size of this region is affected by a numerical factor that can be small or large, depending on the geometry of the submerged body. Since this is an important point for practical application, it will be analysed later. From (2.5) and (2.20) the potential $\Phi(x,y,z,t)$ can be written as

$$\begin{aligned}\Phi(x,y,z,t) = {}& \delta^{\frac{2}{3}}\{\tfrac{1}{2}A(X,T)\,T(y,z)\,\mathrm{e}^{\mathrm{i}(\bar{K}_{\mathrm{T}}x-\bar{\Omega}t)} + (*)\} + \delta\cdot\Phi_{\mathrm{L}}(x,y,z,t)\\ &+ \delta^{\frac{4}{3}}\{\tfrac{1}{4}\mathrm{i}A^2(X,T)\,B_{22}(y,z)\,\mathrm{e}^{2\mathrm{i}(\bar{K}_{\mathrm{T}}x-\bar{\Omega}t)} + (*)\} + \delta^{\frac{5}{3}}\Phi_{\mathrm{N}}(x,y,z,t)\\ &+ \delta^2\{\phi_{21}(y,z;\epsilon x)\,\mathrm{e}^{\mathrm{i}(\bar{K}_{\mathrm{T}}x-\bar{\Omega}t)} + (*)\} + \ldots + O(\delta^{\frac{7}{3}}).\end{aligned} \tag{3.2}$$

The two first terms in (3.2) are, respectively, the leading-order term (2.20b) and the linear solution (2.5). The last one, of order δ^2, is just the potential forced by $q(y;\epsilon x)$, see (2.12); and ... indicates terms of order δ^2 associated with the sum frequency $\omega_1+\omega_2$ of the incoming waves. They are not relevant for the present analysis and the two remaining terms in (3.2) will be explained next. First, however, it is worth pointing out the following: since our interest is restricted to the resonant order δ^2, the function $A(X,T)$ can be considered a constant when the quadratic and cubic operators are applied to it. The discrepancy will appear at order $\delta^{\frac{8}{3}} \ll \delta^2$ and, for the same reason, $\{\bar{K}_{\mathrm{T}}; \bar{\Omega}\}$ can be considered equal to $\{K_{\mathrm{T}}; \Omega\}$ in similar expressions.

The quadratic operator applied to the leading-order term gives rise to two contributions at order $\delta^{\frac{4}{3}}$: one with the form $\frac{1}{4}|A|^2 b_{20}(y)$ and the other given by

$$[\tfrac{1}{4}\mathrm{i}A^2(X,T)\,b_{22}(y)\,\mathrm{e}^{2\mathrm{i}(\bar{K}_{\mathrm{T}}x-\bar{\Omega}t)} + (*)].$$

It is an easy task to check that $b_{20}(y) = 0$, and

$$b_{22}(y) = \left\{ -3\Omega\lambda_0^2 T^2(y,z) + \Omega\left(\frac{\partial T}{\partial y}(y,z)\right)^2 + \Omega\frac{\partial}{\partial y}\left(T(y,z)\frac{\partial T}{\partial y}(y,z)\right)\right\}_{z=0}. \quad (3.3)$$

The wavenumber λ_0 in (3.3) is defined in (1.1) and the potential $B_{22}(y,z)$ in (3.2) is excited by $b_{22}(y)$ at the free surface. The term $\delta^{\frac{5}{3}}\Phi_N$ in (3.2) comes from the quadratic interaction between the leading-order term and the linear solution $\delta \cdot \Phi_L$ and only its quadratic interaction with $\delta \cdot \Phi_L$, of order $\delta^{\frac{8}{3}} \ll \delta^2$, will contain the excitation factor $\exp \mathrm{i}(K_T x - \Omega t)$. It follows that $\delta^{\frac{5}{3}}\Phi_N$ is of no relevance for the present analysis.† The quadratic and cubic operators applied to (3.2) give

$$Q[\Phi;\Phi] + C[\Phi;\Phi;\Phi] = \delta^{\frac{4}{3}}\{\tfrac{1}{4}\mathrm{i}A^2(X,T)\,b_{22}(y)\,\mathrm{e}^{2\mathrm{i}(\bar{K}_T x - \bar{\Omega} t)} + (*)\}$$
$$+\delta^2\{[-q(y;\epsilon x) + (n_2(y)+n_3(y))\,|A(X,T)|^2 A(X,T)]\,\mathrm{e}^{\mathrm{i}(\bar{K}_T x - \bar{\Omega} t)} + (*)\} + \dots. \quad (3.4)$$

The functions $\{q(y,\epsilon x); b_{22}(y)\}$ are defined in (2.10), (3.3); and $\{n_2(y); n_3(y)\}$ come, respectively, from the quadratic interaction between the leading-order term and the potential $B_{22}(y,z)$ and from the cubic interaction of the leading-order term. They are expressed as

$$n_2(y) = -\frac{1}{8}\left\{6\Omega\lambda_0^2 T(y,0)\,B_{22}(y,0) - 2\Omega^3 T(y,0)\,b_{22}(y)\right.$$
$$+\Omega\left(\frac{\partial T}{\partial y}(y,0)\frac{\partial B_{22}}{\partial y}(y,0)\right) - \Omega\frac{\partial}{\partial y}\left(T(y,0)\frac{\partial B_{22}}{\partial y}(y,0)\right)$$
$$\left.+2\Omega\frac{\partial}{\partial y}\left(B_{22}(y,0)\frac{\partial T}{\partial y}(y,0)\right)\right\}, \quad (3.5a)$$

and

$$n_3(y) = -\frac{1}{16}\left\{-(3K_T^2(K_T^2 - \tfrac{5}{3}\Omega^4) - \Omega^8)\,T^3(y,0) + (16\Omega^4 + 3K_T^2)\right.$$
$$\left.\times T(y,0)\left(\frac{\partial T}{\partial y}(y,0)\right)^2 + 3\left(\frac{\partial T}{\partial y}(y,0)\right)^2\frac{\partial^2 T}{\partial y^2}(y,0)\right\}. \quad (3.5b)$$

Placing (3.2) and (3.4) into (2.6), and separating terms of like orders in δ, one obtains

(*a*) *order* $\delta^{\frac{4}{3}}$:

$$\nabla^2 B_{22} - (2K_T)^2 B_{22} = 0, \quad (3.6a)$$

$$\frac{\partial B_{22}}{\partial z}(y,0) = (2\Omega)^2 B_{22}(y,0) + b_{22}(y), \quad (3.6b)$$

$$\nabla B_{22}\cdot \boldsymbol{n}|_{\partial B} = 0, \quad (3.6c)$$

$$\frac{\partial B_{22}}{\partial z}(y,-h) = 0, \quad (3.6d)$$

where $B_{22}(y,z)$ satisfies the radiation condition

$$B_{22}(y,z) \sim B_{22}^{\pm}\mathrm{e}^{\mathrm{i}D_0(\Omega)\cdot|y|}f_0(z;2\Omega), \quad y \to \pm\infty. \quad (3.6e)$$

† It shows, however, that a second slow scale $(X_1, T_1) = \delta^{\frac{2}{3}}(X,T)$ should appear at higher order. The present analysis is valid then for $(X;T) < O(\delta^{-\frac{2}{3}})$.

The function $f_0(z; 2\Omega)$† is defined in (3.2) of I, with 2Ω in place of Ω (notice that $\mathrm{d}f_0(0; 2\Omega)/\mathrm{d}z = (2\Omega)^2 f(0; 2\Omega)$), and a full explanation of the above condition will be given after the order-δ^2 equations are written.

(*b*) *order* δ^2:

$$\nabla^2\phi_{21} - K_{\mathrm{T}}^2\phi_{21} = -\left[\mathrm{i}K_{\mathrm{T}}\frac{\partial A}{\partial X} - \Delta K_{\mathrm{T}} K_{\mathrm{T}} A\right] T(y,z), \tag{3.7a}$$

$$\frac{\partial\phi_{21}}{\partial z}(y,0) - \Omega^2\phi_{21}(y,0) = \left[\mathrm{i}\Omega\frac{\partial A}{\partial T} + \Delta\Omega\Omega A\right] T(y,0) + (n_2(y)+n_3(y))\,|A|^2 A - q(y;\epsilon x), \tag{3.7b}$$

$$\nabla\phi_{21}\cdot\boldsymbol{n}|_{\partial B} = 0, \tag{3.7c}$$

$$\frac{\partial\phi_{21}}{\partial z}(y,-h) = 0, \tag{3.7d}$$

$$\phi_{21} \text{ bounded at infinity.} \tag{3.7e}$$

Condition (3.7*e*) is sufficient for the present analysis, as will be seen later. In the following, the structure of (3.6) will be briefly analysed.

It should first be observed that, from the inequality $2K_{\mathrm{T}}(\Omega)/K_{\mathrm{T}}(2\Omega) < 1$ (see (4.3) in I), the trapped mode at frequency 2Ω is not excited in (3.6). So this set of equations is well behaved, and since $b_{22}(y) \to 0$ when $y \to \infty$, the behaviour of $B_{22}(y,z)$ at infinity is given by (3.6*e*), with

$$D_0(\Omega) = 4K_0\left[1 - \frac{1}{4}\left(\frac{K_{\mathrm{T}}}{K_0}\right)^2\right]^{\frac{1}{2}}, \tag{3.8}$$

where $\{K_{\mathrm{T}}; K_0\}$ are defined in (1.1). It should be observed that $D(\Omega)$ is real if $K_{\mathrm{T}}/K_0 < 2$ – a condition usually satisfied by trapped modes – and in this circumstance $B_{22}(y,z)$ radiates energy to infinity. Since this potential is solely related to the trapped mode, then (3.6*e*) indicates that this mode actually leaks energy to infinity at higher order, when $K_{\mathrm{T}}/K_0 < 2$. The radiated power P_{r} can be obtained from (3.6*e*) if the pressure is multiplied by the horizontal velocity and then integrated along the depth. It is given by

$$P_{\mathrm{r}} = \tfrac{1}{4}\Omega\left[\operatorname{Im}\int_{-\infty}^{\infty} B_{22}(y,0)\,b_{22}(y)\,\mathrm{d}y\right]\delta^{\frac{8}{3}}|A|^4, \tag{3.9}$$

where the relation between the excitation term $b_{22}(y)$ and the far-field amplitude $B_{22}^{\pm}$ has been obtained from the energy integral of (3.6). If $K_{\mathrm{T}}/K_0 > 2$ then $\mathrm{i}D_0(\Omega) < 0$ and $B_{22}(y,z)$ is exponentially decaying as $|y| \to \infty$. In this case $\operatorname{Im} B_{22}(y,0) \equiv 0$ and the radiated power is obviously zero. More will be said about the influence of $B_{22}(y,z)$ later.

The analysis of (3.7) is relatively simple at this point. If the field equation is multiplied by $T(y,z)$ and integrated by parts in A_∞, the following identity is obtained:

$$\begin{aligned} M_\infty(\phi_{21},T) = {} & \mathrm{i}\left(I_0(\Omega)\frac{\partial A}{\partial T} + I_1(\Omega)\frac{\partial A}{\partial X}\right) + (I_0(\Omega)\,\Delta\Omega - I_1(\Omega)\,\Delta K_{\mathrm{T}})\,A \\ & + \left(\int_{-\infty}^{\infty}(n_2(y)+n_3(y))\,T(y,0)\,\mathrm{d}y\right)|A|^2 A - I_0(\Omega)\,Q_{\mathrm{L}}(\epsilon x). \end{aligned} \tag{3.10}$$

† To make the analysis more general, arbitrary water depth has been assumed here. For infinite water depth, $f_0(z; 2\Omega) = (8K_0)^{\frac{1}{2}}\,\mathrm{e}^{4K_0 z}$; $K_0 = \Omega^2$.

The parameters $I_0(\Omega)$, etc., have been defined in (1.3), the bilinear form $M_\infty(.;.)$ in (2.14), and $Q_{\mathrm{L}}(\epsilon x)$ in (2.16). Since $M_\infty(\phi_{21};T)=0$, from (3.10) the following equation can be derived for $A(X,T)$:

$$\left.\begin{aligned} \mathrm{i}\left(\frac{\partial A}{\partial T}+c\frac{\partial A}{\partial X}\right)+(\sigma+\mathrm{i}\mu_{\mathrm{v}})\,A+(n+\mathrm{i}\mu_{\mathrm{r}})\,|A|^2A &= Q_{\mathrm{L}}(\gamma X),\\ \gamma &= \epsilon/\delta^{\frac{4}{3}}. \end{aligned}\right\} \tag{3.11}$$

In (3.11) the wave velocity c is defined in (1.4) and the *detuning parameter* σ is given by

$$\sigma = \Delta\Omega - c\Delta K_{\mathrm{T}}. \tag{3.12}$$

The parameter μ_{v} is the viscous damping coefficient and an expression for it will be derived in the next section. The coefficient $n+\mathrm{i}\mu_{\mathrm{r}}$ is defined by the expression

$$n+\mathrm{i}\mu_{\mathrm{r}} = \frac{1}{I_0(\Omega)}\int_{-\infty}^{\infty}(n_2(y)+n_3(y))\,T(y,0)\,\mathrm{d}y, \tag{3.13}$$

where, using (3.5), one obtains

$$n(\Omega) = \frac{1}{4I_0(\Omega)}\,\mathrm{Re}\left(\int_{-\infty}^{\infty}B_{22}(y,0)\,b_{22}(y)\,\mathrm{d}y\right)-\frac{1}{16}\left\{\Omega^8\left(3+\right.\right.$$
$$\left.\left.+11\left(\frac{\lambda_0}{K_0}\right)^2-3\left(\frac{\lambda_0}{K_0}\right)^4\right)\frac{I_2(\Omega)}{I_0(\Omega)}+23\Omega^4\left(1+3\left(\frac{\lambda_0}{K_0}\right)^2\right)\frac{I_3(\Omega)}{I_0(\Omega)}-\frac{I_4(\Omega)}{I_0(\Omega)}\right\}, \tag{3.14a}$$

$$\mu_{\mathrm{r}}(\Omega) = \frac{1}{4I_0(\Omega)}\,\mathrm{Im}\left(\int_{-\infty}^{\infty}B_{22}(y,0)\,b_{22}(y)\,\mathrm{d}y\right)\geqslant 0. \tag{3.14b}$$

Notice that the influence of $B_{22}(y,z)$† appears only in the form

$$\int_{-\infty}^{\infty}B_{22}(y,0)\,b_{22}(y)\,\mathrm{d}y$$

and this quantity can be determined by means of a variational approximation, in a manner similar to the one indicated in Aranha & Pesce (1988). In this context the actual computation of the influence of $B_{22}(y,z)$ is not difficult from a numerical point of view; see Appendix for details.

The parameter γ in (3.11) is the ratio between two slow length scales: one, ϵx, associated with the body slenderness, and the other $\delta^{\frac{4}{3}}x$, associated with the nonlinear resonance. It is important, however, to keep in mind the following point: the numerical value of γ can be large or small but, from a theoretical point of view, it must remain constant as $(\epsilon;\delta)\rightarrow 0$. The theory presented here can deal, then, with the longitudinal variation of the linear scattered potential $\phi_{\mathrm{s},j}(y,z;\epsilon x)$ but arguments to be given next show that this dependence is, in general, of higher order and can be disregarded. Indeed for long waves ($\omega_j^2 L\sim O(1)$; $\omega_j^2\sim O(\epsilon)$) the scattered potential is of relative order ϵ, and can be ignored consistently. For short waves ($\omega_j^2 B\sim O(1)$; $\omega_j^2\sim O(1)$) and $\sin\alpha_j\sim O(1)$, the scattered potential can be represented by a strip-theory approximation, with a relative error of order ϵ. As is well known, the strip-

† The terms proportional to $B_{22}(y,z)$ in (3.5a) can be integrated by parts in (3.13) and the above expressions are then obtained with the help of (3.3). From the energy integral of (3.6) it can be seen that $\mu_{\mathrm{r}}\geqslant 0$.

theory approximation for the scattered wave is independent of x for a body with uniform cross-section, and so the x-dependence of Q_L is again of higher order. The only case where the x-dependence is of relevance is when one of the incident waves is in the head-sea direction and the associate wavelength is short. For a semisubmersible platform, however, the x-dependence can be ignored even in this case to a first approximation, since these structures are relatively transparent to the action of sea waves and the effect of the scattered potential is small. In all these cases one can take consistently

$$Q_L(\epsilon x) = Q_L[1+O(\epsilon)], \quad Q_L = \text{constant}, \tag{3.15}$$

which certainly simplifies some aspects of the analysis. Besides the exciting term Q_L and 'group velocity' $c(\Omega)$, the wave equation (3.11) depends on four parameters: the damping factors $\{\mu_v; \mu_r\}$, the nonlinear restoring coefficient n and the detuning σ.

The physical meaning of the damping factors will be explained in the next section, but it is important to observe here a particular feature of the detuning parameter σ. In fact, as is clear from (3.12), σ can be zero even for non-zero values of $\{\Delta\Omega; \Delta K_T\}$ – it suffices that $\Delta\Omega/\Delta K_T = c(\Omega)$. This result means only that the exciting waves are then tuned to the trapped mode $\{K_T(\Omega+\Delta\Omega\delta^{\frac{4}{3}}); T(y,z;\Omega+\Delta\Omega\delta^{\frac{4}{3}})\}$, since $c(\Omega) = \mathrm{d}\Omega/\mathrm{d}K_T = \Delta\Omega/\Delta K_T$ in the limit $\delta \to 0$. Notice that the discrepancy between $T(y,z;\Omega)$ and $T(y,z;\Omega+\Delta\Omega\delta^{\frac{4}{3}})$ is of order $\delta^{\frac{4}{3}}$ and so (3.11) can be consistently used to analyse the trapped mode at frequency $\Omega+\Delta\Omega\delta^{\frac{4}{3}}$.

4. Energy equation and viscous damping

To obtain a clear physical meaning for the damping factors $\{\mu_v; \mu_r\}$ and, simultaneously, to check the consistency of the wave equation (3.11), it is worthwhile looking for the energy equation associated with the trapped-mode excitation.

It is convenient first to obtain the energy equation directly from physical arguments. In this way the average kinetic energy in one cycle is given by

$$\tilde{E}_c = 0.5\rho B^3 g \iint_{A_\infty} \langle \hat{\nabla}\Phi \rangle^2 \,\mathrm{d}A_\infty,$$

where a tilde is used for dimensional variables and $\langle\cdot\rangle$ is the average operator. From (1.3) and (3.2), however

$$\iint \langle \hat{\nabla}\Phi \rangle^2 \,\mathrm{d}A_\infty = 0.5\delta^{\frac{4}{3}}|A|^2\Omega I_0(\Omega),$$

to leading order. In spite of nonlinearity the potential energy is, to this order, equal to the kinetic energy†, and then

$$\tilde{E} = \rho g B^3 \delta^{\frac{4}{3}} \Omega I_0(\Omega) \tfrac{1}{2} |A(X,T)|^2. \tag{4.1}$$

The trapped wave is excited by a pressure in the free surface caused by the nonlinear interaction of the incoming waves. In dimensional variables,

$$\left.\begin{aligned} \frac{1}{g}\frac{\partial^2\tilde{\Phi}}{\partial \tilde{t}^2} + \left.\frac{\partial\tilde{\Phi}}{\partial\tilde{z}}\right|_{\tilde{z}=0} &= -\frac{1}{\rho g}\frac{\partial\tilde{p}_e}{\partial\tilde{t}}(\tilde{x},\tilde{y},\tilde{t}), \\ \tilde{p}_e(\tilde{x},\tilde{y},\tilde{t}) &= \rho g B p_e(x,y,t), \\ \frac{\partial p_e}{\partial t}(x,y,t) &= \delta^2[q(y;\epsilon x)\,\mathrm{e}^{\mathrm{i}(\bar{K}_T x - \Omega t)} + (*)], \end{aligned}\right\} \tag{4.2}$$

† To leading order the potential Φ follows the trapped wave that, being a self-sustained oscillation, must have the potential energy balanced by kinetic energy (see (2.14) with $\Psi = T$).

where this last expression comes from (2.9). The average power induced by this pressure is given by

$$\tilde{P}_e = \int_{\infty}^{\infty} \left\langle \rho g B p_e(x,y,t) \frac{\partial \tilde{\Phi}}{\partial \tilde{z}}(\tilde{x},\tilde{y},\tilde{0},\tilde{t}) \right\rangle d\tilde{y}.$$

Using (2.2), (2.16), (3.2) and (4.2), one obtains

$$\tilde{P}_e = \frac{\rho g B^3}{(B/g)^{\frac{1}{2}}} \delta^{\frac{8}{3}} \Omega I_0(\Omega) \tfrac{1}{2}(-\mathrm{i}Q_L A^* + \mathrm{i}Q_L^* A). \tag{4.3}$$

From (3.9) and (3.14*b*) the trapped-wave radiated power is given by

$$\tilde{P}_r = \frac{\rho g B^3}{(B/g)^{\frac{1}{2}}} \delta^{\frac{8}{3}} \Omega I_0(\Omega)\, \mu_r |A(X,T)|^4. \tag{4.4}$$

The power dissipated by viscosity will now be estimated. Since $\delta^{\frac{2}{3}} \ll 1$, there is no flow separation and in the periodic boundary layer, the convective part of the fluid acceleration can be disregarded in relation to $\partial\tilde{u}/\partial\tilde{t}$. Thus, if $\tilde{U}_0(s)\cos\tilde{\Omega}\tilde{t}$ is the potential flow outside the layer, then the velocity profile is given by (see Schlichting 1968, p. 411)

$$\tilde{u}_0(\tilde{s},\tilde{\eta},\tilde{t}) = \tilde{U}_0(\tilde{s})[\cos\tilde{\Omega}\tilde{t} - \exp(-\eta/\surd 2)\cos(\tilde{\Omega}\tilde{t} - \eta/\surd 2)] \tag{4.5a}$$

$$\eta = (\tilde{\Omega}/\nu)^{\frac{1}{2}}\tilde{\eta}, \tag{4.5b}$$

where ν is the kinematic viscosity and $e = (\nu/\tilde{\Omega})^{\frac{1}{2}}$, the order of the boundary-layer thickness. The power dissipated in the boundary layer BL is then

$$\tilde{P}_v = \left\langle \rho\nu \iint_{BL} \left(\frac{\partial \tilde{u}_0}{\partial \tilde{\eta}}\right)^2 dA_{BL} \right\rangle = \rho\nu \left(\frac{\tilde{\Omega}}{\nu}\right)^{\frac{1}{2}} \left\langle \int_0^\infty d\eta \int_{\partial B} \left(\frac{\partial \tilde{u}_0}{\partial \eta}\right)^2 d\,\partial B \right\rangle. \tag{4.5c}$$

Using (3.2) and (4.5*a*), the last integral in (4.5*c*) can be expressed as

$$\left\langle \int_0^\infty d\eta \int_{\partial B} \left(\frac{\partial \tilde{u}_0}{\partial \eta}\right)^2 d\,\partial B \right\rangle = \surd 2 \left(\int_0^\infty \xi^2\, e^{-\xi}\, d\xi\right) B^2 g \int_{\partial B} U_0^2(s)\, ds, \tag{4.5d}$$

where $U_0^2(s) = 0.5\delta^{\frac{4}{3}}|A(X,T)|^2[(\partial T/\partial s)^2 + K_T^2 T^2(s)]$. Since $\tilde{\Omega} = \Omega(g/B)^{\frac{1}{2}}$ then, with the help of (1.3), the following expression is obtained for $\tilde{P}_v$:

$$\left.\begin{aligned} \tilde{P}_v &= \frac{\rho g B^3}{(B/g)^{\frac{1}{2}}} \delta^{\frac{8}{3}} \Omega I_0(\Omega)\, \mu_v |A(X,T)|^2, \\ \mu_v &= \frac{1}{\delta^{\frac{4}{3}}} \left(\frac{2(B/g)^{\frac{1}{2}}}{B^2}\right)^{\frac{1}{2}} \left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{I_5(\Omega)}{I_0(\Omega)}. \end{aligned}\right\} \tag{4.6}$$

The above formula for the viscous damping coefficient μ_v is only of academic interest if the body has a sharp corner, since then flow separation is unavoidable. Even in this case, however, the effect of viscosity may perhaps be simulated by the term $\mathrm{i}\mu_v$ in (3.11), with a proper choice for μ_v.

The trapped-wave group velocity is $c = d\Omega/dK$, and since these waves propagate in the x-direction, energy conservation implies

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\frac{\tilde{E}}{B^{\frac{1}{2}}/g} = \tilde{P}_e - \tilde{P}_v - \tilde{P}_r. \tag{4.7}$$

Using (4.1), (4.3), (4.4) and (4.6) in (4.7), one obtains

$$\left(\frac{\partial}{\partial T}+c\frac{\partial}{\partial X}\right)(\tfrac{1}{2}|A|^2)+\mu_v|A|^2+\mu_r|A|^4=\tfrac{1}{2}(-\mathrm{i}Q_L A^*+\mathrm{i}Q_L^* A). \tag{4.8}$$

Equation (4.8), obtained here by direct physical arguments, is just the energy identity of the nonlinear wave equation (3.11). This fact not only checks the consistency of (3.11), but it also provides a clear physical meaning for the damping factors $\{\mu_v;\mu_r\}$.

In a two-dimensional wave tank, A does not depend on X. If the wavemaker is switched off at $T=0$, then $Q_L=0$ for $T>0$ and the energy equation (4.8) reads

$$\frac{\mathrm{d}}{\mathrm{d}T}|A|^2=-2(\mu_v+\mu_r|A|^2)|A|^2\geqslant-2(\mu_v+\mu_r|A_0|^2)|A|^2, \tag{4.9a}$$

where A_0 is the wave amplitude at $T=0$. Since $T=\delta^{\frac{4}{3}}t$ it follows from (4.9*a*) that

$$\frac{|A(t)|^2}{|A_0|^2}\geqslant\exp\left(-2(\mu_v+\mu_r|A_0|^2)\delta^{\frac{4}{3}}t\right). \tag{4.9b}$$

This expression shows that the decay rate of the trapped mode is very small, a conclusion in accordance with the laboratory observations described in McIver & Evans (1985) and also the introduction of I. For a slender three-dimensional cylinder, (4.8) is a wave equation and the same conclusion does not apply. In this case the perturbation is swept away with the group velocity $c(\Omega)$ when the 'wavemaker' is switched off.

5. Normalization of the wave equation, and discussion

As explained in §3 it is convenient, at least in a first analysis†, to use the approximation (3.15),

$$Q_L(\epsilon x)=|Q_L|\,\mathrm{e}^{\mathrm{i}\Psi}H(X), \tag{5.1}$$

where both $|Q_L|$ and Ψ are X-independent and $H(X)$ is equal to 1 for $0\leqslant X\leqslant 1/\gamma$ and zero otherwise. For $\mu_r=\mu_v=0$ the solution $A(X,T)$ of (3.11) is of order $|A|^3\sim O(Q_L/n)$ when $\sigma=0$ and, in this way, the following normalized variables are introduced:

$$\left.\begin{aligned}\bar{X}=(|Q_L|^2|n|)^{\frac{1}{3}}X,\quad \bar{T}=(|Q_L|^2|n|)^{\frac{1}{3}}T,\quad \bar{\sigma}=(|Q_L|^2|n|)^{-\frac{1}{3}}\sigma,\\ \bar{\mu}_v=(|Q_L|^2|n|)^{-\frac{1}{3}}\mu_v,\quad \bar{\mu}_r=|n|^{-1}\mu_r,\quad \bar{A}(\bar{X},\bar{T})=(|n|\,|Q_L|^{-1})^{\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\Psi}A(X,T).\end{aligned}\right\} \tag{5.2}$$

In terms of (5.2) the wave equation (3.11) can be written as

$$\mathrm{i}\left(\frac{\partial\bar{A}}{\partial\bar{T}}+\frac{\partial\bar{A}}{\partial\bar{X}}\right)+(\bar{\sigma}+\mathrm{i}\bar{\mu}_v)\bar{A}+(\operatorname{sgn}(n)+\mathrm{i}\bar{\mu}_r)|\bar{A}|^2\bar{A}=H(\bar{X}), \tag{5.3a}$$

where $\bar{A}(\bar{X},\bar{T})$ must be subjected to boundary and initial conditions, for example

$$\bar{A}(\bar{X},\bar{T})=0;\quad \bar{X}<0\quad \text{any }\bar{T},\quad \bar{A}(\bar{X},0)=0. \tag{5.3b}$$

When $\bar{\sigma}\to\infty$ the solution of (5.3) should approach the linear solution $\bar{A}(\bar{X},\bar{T})=$

† It should be observed, however, that in a vicinity of order $x\sim O(1)\,(X\sim O(\delta^{\frac{4}{3}}))$ from the ends $x=0;\,x=L\,(X=0;\,X=1/\gamma)$ the actual behaviour of $Q_L(\epsilon X)$ cannot be determined by this quasi-two-dimensional theory. This is a common feature of the standard slender body theories.

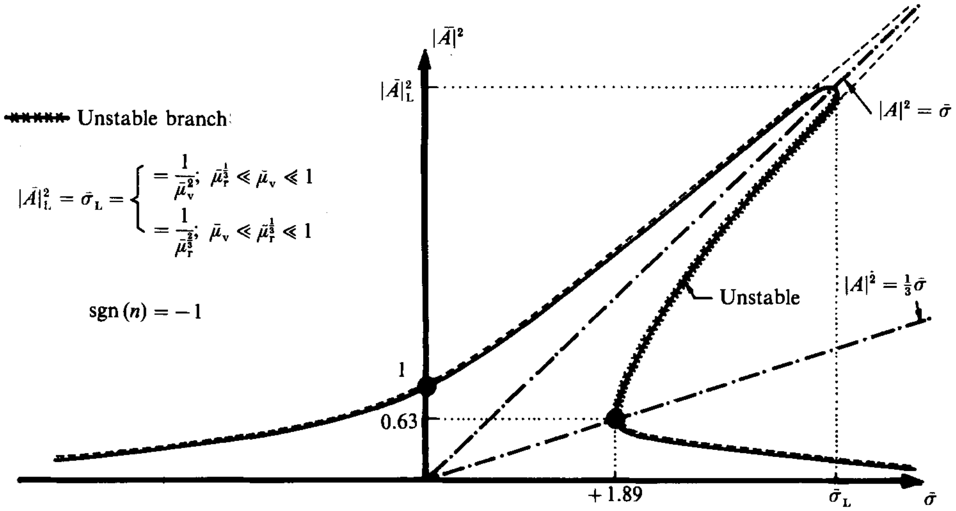

FIGURE 1. The roots of equation (5.3c).

$H(\bar{X})/\bar{\sigma}$ for $\bar{T} \gg 1$ where, obviously, $\bar{A}(\bar{X},\bar{T})$ is then independent of $\bar{X}$ and $\bar{T}$ within the body ($0 < \bar{X} < 1/\bar{\gamma}$). It is known that solutions of some forced nonlinear wave equations do not approach a steady state when $\bar{T} \to \infty$, even when the forcing term is time-independent and localized in space. This particular behaviour, however, seems to be related to the existence of solitons,† and since the present wave equation does not have this class of solutions (it is non-dispersive) a steady state is expected in the limit $\bar{T} \to \infty$. If this is the case, the solution of (5.3a) should be analogous to the resonant response of a nonlinear oscillator, where the amplitude is a root of the algebraic equation

$$(\bar{\sigma} + \mathrm{i}\bar{\mu}_{\mathrm{v}})\bar{A} + (\operatorname{sgn}(n) + \mathrm{i}\bar{\mu}_{\mathrm{r}})|\bar{A}|^2\bar{A} = 1. \tag{5.3c}$$

In particular, a jump-like phenomenon should be expected for (5.3a), and, for future reference, the roots of (5.3c) have been plotted in figure 1 as a function of the normalized detuning $\bar{\sigma}$. Anyway the wave amplitude $\bar{A}$ will be of order 1 when $\bar{\sigma} \sim (1)$, and of order $1/\bar{\sigma} \ll 1$ if $\bar{\sigma} \gg 1$. From (5.2) it follows that the normalized size of the near-resonant region can be gauged by the parameter

$$\Delta\bar{\sigma} = (|Q_{\mathrm{L}}|^2\,|n|)^{\frac{1}{3}}, \tag{5.4a}$$

since the near-resonant region will be large (small) if $\Delta\bar{\sigma}$ is large (small), for the same detuning $\sigma \sim O(1)$. The related wave-amplitude parameter is given by

$$\Delta\bar{A} = (|n|^{-1}\,|Q_{\mathrm{L}}|)^{\frac{1}{3}}, \tag{5.4b}$$

since $A \sim O(\Delta\bar{A})$ whenever $\bar{A} \sim O(1)$. Notice that $\Delta\bar{\sigma}\Delta\bar{A} = |Q_{\mathrm{L}}|$, the intensity of the excitation, and both parameters give a good idea of the importance that a possible trapped-mode excitation may have on the performance of a submerged structure.

The two pontoons of a semisubmersible platform, for example, are typically rectangular cylinders with length $\tilde{L} \approx 100$ m, beam $\tilde{B} \approx 16$ m, width $\tilde{D} \approx 8$ m and

† Even for the cubic Schrödinger equation, a steady state is reached when sgn $(n) = -1$, in which case a soliton-like solution does not exist. In this circumstance a jump-like phenomenon is also observed; see Aranha *et al.* (1982).

distant $\tilde{S} \approx 12$ m from the free surface. In this case $\epsilon = \tilde{B}/\tilde{L} \approx 0.16$ and, with this motivation, a rectangular box with beam $B = 2b = 1.0$, width D and distant S from the free surface will be considered in the following. If S is relatively large the trapped mode can be approximated by (see (4.4) and figure 2 of I)

$$T(y,z;\Omega) = \begin{cases} e^{K_0 z}; & |y| \leqslant b, \\ e^{-\lambda_0(|y|-b)}\,e^{K_0 z}; & |y| \geqslant b, \end{cases} \tag{5.5}$$

where $\lambda_0 = (K_T^2 - K_0^2)^{\frac{1}{2}}$ is then the root of equation (4.5) in I or, in simplified form,

$$\frac{\lambda_0}{K_0} = K_0\,e^{-2K_0 S}(1-e^{-2K_0 D}). \tag{5.6}$$

To obtain (5.6), λ_0/K_0 was assumed small, as in fact it is if S is large, and the quadratic term has been ignored in (4.5) of I. Using (5.5) in (1.3) the integrals $I_0(\Omega), \ldots, I_4(\Omega)$ can be computed, and from (3.14) and (A 17) (see Appendix) a closed-form expression for $n(\Omega)$ and $\mu_r(\Omega)$ can be derived. If then powers of λ_0/K_0 are neglected compared with 1, the following formulas are obtained:

$$n \approx -\frac{3K_0^{\frac{7}{2}}}{32}, \quad \mu_r \sim O\left(\left(\frac{\lambda_0}{K_0}\right)^5\right) \ll 1. \tag{5.7a}$$

It remains to determine Q_L and some further assumptions must be made to render the analysis feasible. First, the incident wave (namely, the Froude–Krilov approximation for diffraction potential) can be used in (2.10), since only the order of magnitude $|Q_L|$ is needed to estimate $\Delta\bar{\sigma}$ and $\Delta\bar{A}$. Second, it will be assumed here that the two incident waves are in the same direction, that is $\alpha = \alpha_1 = \alpha_2$. Using then expression (5.4) of I, one obtains the tuning condition $\cos\alpha = (1+r)^{-1}$ with $r = \omega_1/\Omega$, ω_1 being the frequency of the longest incident wave. In this way a closed-form expression can be also derived for $|Q_L|$, and further simplified if λ_0/K_0 is neglected compared to 1. The final formula is given by

$$\left.\begin{aligned} |Q_L| &= f(K_0 b; r)\frac{\lambda_0}{K_0}, \\ f(K_0 b; r) &= \left(\frac{2r(1+r)}{1+2r}\right)^{\frac{1}{2}}\left[K_0 b\frac{\sin(2r(1+1/r)^{\frac{1}{2}}K_0 b)}{2r(1+1/r)^{\frac{1}{2}}} + \cos(2r(1+1/r)^{\frac{1}{2}}K_0 b)\right]. \end{aligned}\right\} \tag{5.7b}$$

For the range $K_0 = \tilde{K}_0\tilde{B}$ of practical interest the function $f(K_0 b; r)$ is of order 1. For example, taking $\omega_1^2 = 0.25$ (corresponding to $\tilde{T}_1 \approx 16$ s for $\tilde{B} = 16$ m), $f(K_0 b; r)$ changes from 0.94 to 0.64 when K_0 changes from 0.50 to 2.00. Assuming $f(K_0 b; r) = 1.0$ in (5.7b), the following simple expressions for $\Delta\bar{\sigma}$ and $\Delta\bar{A}$ can be derived:

$$\Delta\bar{\sigma} \approx 0.45K_0^{\frac{7}{6}}\left(\frac{\lambda_0}{K_0}\right)^{\frac{2}{3}}, \quad \Delta\bar{A} \approx 2.22K_0^{-\frac{7}{6}}\left(\frac{\lambda_0}{K_0}\right)^{\frac{1}{3}}. \tag{5.8}$$

Notice that (5.8), based on (5.5), (5.6), is essentially correct in the limits $K_0 \rightarrow 0$, $K_0 \rightarrow \infty$ and $S \rightarrow \infty$; see I for details. In this context the above approximation allows one to do some qualitative analysis of the likelihood of trapped-mode excitation. For example; if S is large, λ_0/K_0 will be very small for all frequencies (see (5.6)), and (5.8)

	$S/B = 12/16;\ D/B = 0.5$			$S/B = 3/16;\ D/B = 0.5$		
$K_0 B$	λ_0/K_0	$1/\Delta\bar{\sigma}$	$\Delta\bar{A}$	λ_0/K_0	$1/\Delta\bar{\sigma}$	$\Delta\bar{A}$
0.50	0.09	25.0	2.2	0.16	17.0	2.7
1.00	0.14	8.3	1.2	0.43	3.9	1.7
1.50	0.12	5.9	0.8	0.66	1.8	1.2
2.00	0.09	5.0	0.4	0.82	1.1	0.9

TABLE 1. Values of λ_0/K_0, $\bar{\sigma} \approx 1/\Delta\bar{\sigma}$ and $\Delta\bar{A}$ from (5.6) and (5.8) for a rectangular submerged body with beam B, width D and distant S from the free surface.

indicates that a trapped-mode excitation will be unlikely. Or, in short, the likelihood of trapped mode excitation increases (the size $\Delta\bar{\sigma}$ of the near-resonant region increases) when the body approaches the free surface ($S \rightarrow 0$). Expressions (5.6) and (5.8) also show that the trapped-wave phenomenon is essentially restricted to the frequency range $K_0 = \tilde{K}_0 \tilde{B} \geqslant O(1)$. In fact, from (5.6) it follows that $\lambda_0/K_0 \leqslant K_0$, and from (5.8) $\Delta\bar{\sigma} \leqslant 0.45 K_0^{\frac{11}{6}}$, $\Delta\bar{A} \leqslant 2.22 K_0^{-\frac{5}{6}}$. The amplitude parameter $\Delta\bar{A}$ increases when K_0 decreases, but since $\Delta\bar{\sigma} \leqslant 0.45 K_0^{\frac{11}{6}}$ the near-resonant region is very small when $K_0 < O(1)$. Or, in short, trapped-mode excitation is unlikely if $K_0 < O(1)$. A similar conclusion also holds true if $K_0 > O(1)$ with $K_0 \mathrm{e}^{-2K_0 S} \ll 1$.

If one recalls that $\bar{\sigma} \approx 1/\Delta\bar{\sigma}$ when $\sigma \sim O(1)$ and takes the nonlinear oscillator equation (5.3c) for reference, three roots of this equation exist for $\bar{\sigma} \approx 1/\Delta\bar{\sigma} \geqslant 1.9$: one above (stable), another below (unstable) the line $|\bar{A}|^2 = \bar{\sigma}$, and a third one (stable) below the line $|\bar{A}|^2 = \frac{1}{3}\bar{\sigma}$. In this latter branch the root (5.3c) can be approximated by $|\bar{A}| \approx 1/\bar{\sigma}$, leading to the following estimate: $A \approx \lambda_0/K_0$, see (5.2) and (5.7b), where $\delta^{-\frac{1}{3}}A$ is the relative value of the trapped-wave amplitude compared to the incident wave. In the upper (stable) branch the size of A is determined by $\bar{A}\Delta\bar{A}$, but the theoretical limit $|\bar{A}|_{\mathrm{L}} = \bar{\sigma}_{\mathrm{L}}^{\frac{1}{2}} = 1/\bar{\mu}_{\mathrm{v}}$ should seldom be reached in reality. The upper branch ($\bar{A}^+$) is stable with respect to an infinitesimal perturbation, and for large $\bar{\sigma}$ it gets very close to the unstable branch ($|\bar{A}|^-$) below the line $|\bar{A}|^2 = \bar{\sigma}$. In fact, $(|\bar{A}|^+ - |\bar{A}|^-)/|\bar{A}|^+ \sim \bar{\sigma}^{-\frac{3}{2}}$ for $\bar{\sigma}$ large, and this relation indicates that the upper-branch solution must become unstable with respect to a finite (although small) perturbation long before the theoretical threshold value $|A|_{\mathrm{L}} = \bar{\sigma}_{\mathrm{L}}^{\frac{1}{2}} = 1/\mu_{\mathrm{v}}$ is reached. A jump from the upper to the lower stable branch should occur somewhere in the interval $4 \lesssim \bar{\sigma} = 1/\Delta\bar{\sigma} \lesssim 10$, depending on the intensity of the existing background noise.

Table 1 displays some values of λ_0/K_0, $\bar{\sigma} = 1/\Delta\bar{\sigma}$ and $\Delta\bar{A}$ obtained from (5.6) and (5.8), for two different ratios of S/B. The first case ($S/B = 12/16$; $D/B = 0.5$) corresponds to the pontoon of a typical semisubmersible platform, and the other one to the same geometry placed closer to the free surface.

For $\sigma \sim O(1)$ the normalized detuning $\bar{\sigma} \approx 1/\Delta\bar{\sigma}$ is relatively large when $S/B = 12/16$, and the relative trapped-wave amplitude should be of order $\delta^{-\frac{1}{3}}(\lambda_0/K_0)$, associated with a root in the lower stable branch of figure 1. In this case trapped-mode excitation is not a very strong phenomenon, and it should be restricted to frequencies around the value $K_0 B = 1.0$. For $S/B = 3/16$ the trapped-mode excitation is much stronger and the related wave amplitude should be determined by $\Delta\bar{A}$ when $K_0 B \geqslant 1.0$. In this case also the maximum response is associated with frequencies around the value $K_0 B = 1.0$. In reality the values presented in table 1 tend to underestimate the actual importance of trapped-mode excitation when

$S/B = 3/16$. On the one hand the approximation (5.6) gets worse as S/B decreases and tends† to be a lower bound for λ_0/K_0; on the other hand, powers of λ_0/K_0 that have been neglected in (3.14*a*) tend to increase the value of n and so of $\Delta\bar{\sigma}$, see (5.4*a*).

In general trapped-mode excitation seems to be a relevant phenomenon for submerged bodies, but before the present theory is improved, it would be worthwhile to experimentally confirm the gross features described here.

Appendix – variational approximation for $I_6(\Omega) = \int_{\infty}^{\infty} B_{22}(y,0)\, b_{22}(y)\,\mathrm{d}y$

To compute $I_6(\Omega)$, the potential $B_{22}(y,z)$, the solution of (3.6), must be determined. The purpose of this Appendix is to show that a variational method can be used in such a way that a relatively rough approximation for $B_{22}(y,z)$ provides a much better result for I_6. The procedure is similar to the Rayleigh quotient in vibration problems, where a convenient choice of a trial function provides a good approximation for the eigenvalue.

Details can be found in Aranha & Pesce (1988), who applied the same method to solve the linear diffraction and radiation problems and, to make more general the present analysis, the water depth will be assumed arbitrary in the following. Let then $\{\bar{f}_n(z); n = 0, 1, 2, \ldots\}$ be the orthonormal set given in (3.2) of I, but defined at frequency 2Ω instead of Ω. In an analogous way one can determine the related wavenumbers $\{\bar{K}_0(2\Omega); \bar{\chi}_n(2\Omega)\}$ and so define the coefficients

$$D_0 = (\bar{K}_0^2 - (2K_T)^2)^{\frac{1}{2}}, \quad D_n = (\bar{\chi}_n^2 + (2K_T)^2)^{\frac{1}{2}}. \tag{A 1}$$

Notice that (A 1) reduces to (3.8) if the deep-water dispersion relation $\bar{K}_0 = (2\Omega)^2 = 4K_0$ is used. To make short the exposition the body will be assumed symmetric with relation to the z-axis and so only the region $y > 0$ need be considered ($\{B_{22}(y, z); b_{22}(y)\}$ are even (odd) in y if $T(y,z)$ is even (odd)). If b is the maximum half-beam of the body let $\bar{L}_n(\cdot)$ be the linear functional

$$\bar{L}_n(\Psi) = \int_{-h}^{0} \Psi(b,z) \bar{f}_n(z)\,\mathrm{d}z. \tag{A 2}$$

The general solution of (3.6) for $y \geqslant b$ can be written as a sum of a particular solution $B_{22}^{(\mathrm{p})}(y,z)$ and a homogeneous one. Imposing the constraint $B_{22}^{(\mathrm{p})}(b,z) = 0$ then, from (A 2), it follows that, for $y \geqslant b$,

$$\left.\begin{aligned} B_{22}(y,z) &= B_{22}^{(\mathrm{p})}(y,z) + \bar{B}_0\, \mathrm{e}^{\mathrm{i}D_0(|y|-b)} \bar{f}_0(z) + \sum_{n=1}^{\infty} \bar{L}_n(B_{22})\, \mathrm{e}^{-D_n(|y|-b)} \bar{f}_n(z), \\ \bar{B}_0 &= \bar{L}_0(B_{22}). \end{aligned}\right\} \tag{A 3}$$

If $\bar{A}_\infty$ is the fluid region $y > b$ and $\Psi(y,z)$ is an arbitrary function, the identity below can be easily obtained from (3.6):

$$\iint_{A_\infty} [\nabla B_{22}^{(\mathrm{p})} \cdot \nabla \Psi + (2K_\mathrm{T})^2 B_{22}^{(\mathrm{p})}\, \Psi]\,\mathrm{d}\bar{A}_\infty - (2\Omega)^2 \int_b^\infty B_{22}^{(\mathrm{p})}(y,0)\, \Psi(y,0)\,\mathrm{d}y$$
$$+ \int_{-h}^{0} \left[\frac{\partial B_{22}^{(\mathrm{p})}}{\partial y}(b,z)\, \Psi(b,z) - \frac{\partial B_{22}^{(\mathrm{p})}}{\partial y}(\infty,z)\, \Psi(\infty,z) \right] \mathrm{d}z = \int_b^\infty b_{22}(y)\, \Psi(y,0)\,\mathrm{d}y. \tag{A 4}$$

† It cannot be proven that (5.6) is a lower bound since it is only an approximation for the quadratic equation (4.5) in I. The root of this latter equation is always a lower bound.

Writing $B_{22}^{(\mathrm{p})}(y,z)$ in the form

$$B_{22}^{(\mathrm{p})}(y,z) = \sum_{n=0}^{\infty} A_n(y)\bar{f}_n(z), \qquad \text{(A 5}a\text{)}$$

placing (A 5a) into (A 4) and taking $\Psi(y,z) = \Psi_n(y)\bar{f}_n(z)$, with $\Psi_n(y)$ arbitrary and $n = 0, 1, 2, \ldots$, a second-order ordinary equation for $A_n(y)$ can be derived. Imposing the conditions $A_n(b) = 0$, $\mathrm{d}A_n/\mathrm{d}y \sim C_n A_n$ when $y \to \infty$, one obtains

$$\left.\begin{aligned} A_0(y) &= \bar{f}_0(0)\,\mathrm{e}^{\mathrm{i}D_0 y}\int_b^y \mathrm{d}\zeta\,\mathrm{e}^{-2\mathrm{i}D_0\zeta}\int_\zeta^\infty b_{22}(\eta)\,\mathrm{e}^{\mathrm{i}D_0\eta}\,\mathrm{d}\eta,\\ A_n(y) &= \bar{f}_n(0)\,\mathrm{e}^{-D_n y}\int_b^y \mathrm{d}\zeta\,\mathrm{e}^{2D_n\zeta}\int_\zeta^\infty b_{22}(\eta)\,\mathrm{e}^{-D_n\eta}. \end{aligned}\right\} \qquad \text{(A 5}b\text{)}$$

Since $b_{22}(y) \sim O(\mathrm{e}^{-2\lambda_0 y})$ when $y \to \infty$ (see (1.1) and (3.3)) the following relations can be derived:

$$A_0(y) \sim B_0\,\mathrm{e}^{\mathrm{i}D_0 y}, \quad A_n(y) \sim B_n\,\mathrm{e}^{-D_n y}; \quad n > 0, \qquad \text{(A 6)}$$

in the limit $y \to \infty$. The particular solution is entirely defined in terms of the known function $b_{22}(y)$ and it remains to determine $B_{22}(y,z)$ in the region $A(y \leqslant b)$; see also (A 3). In order to do so it should be observed, from (A 3), that

$$\frac{\partial B_{22}}{\partial y}(b,z) = \frac{\partial B_{22}^{(\mathrm{p})}}{\partial y}(b,z) + \mathrm{i}D_0\bar{B}_0\bar{f}_0(z) - \sum_{n=1}^{\infty} D_n L_n(B_{22})\bar{f}_n(z). \qquad \text{(A 7)}$$

The function $B_{22}(y,z)$ satisfies (3.6a–d) in the fluid region $A(y \leqslant b)$ and the boundary condition (A 7) on the line $y = b$. If (3.6a) is multiplied by an arbitrary $\Psi(y,z)$ and integrated by parts in A the following weak equation is obtained:

$$G(B_{22};\Psi) = \mathrm{i}D_0\bar{B}_0\bar{L}_0(\Psi) + V_1(\Psi), \qquad \text{(A 8)}$$

with

$$\left.\begin{aligned} G(\phi;\Psi) &= \iint_A [\nabla\phi\cdot\nabla\Psi + (2K_{\mathrm{T}})^2\phi\Psi]\,\mathrm{d}A - (2\Omega)^2\int_0^b \phi(y,0)\,\Psi(y,0)\,\mathrm{d}y\\ &\qquad + \sum_{n=1}^{\infty} D_n\bar{L}_n(\phi)\,\bar{L}_n(\Psi),\\ V_1(\Psi) &= \int_0^b b_{22}(y)\,\Psi(y,0)\,\mathrm{d}y + \int_{-h}^0 \frac{\partial B_{22}^{(\mathrm{p})}}{\partial y}(b,z)\,\Psi(b,z)\,\mathrm{d}z. \end{aligned}\right\} \qquad \text{(A 9)}$$

It is not difficult to relate I_6 to the linear functional $V_1(\cdot)$. In fact, from (A 4) and a similar expression for $B_{22}(y,z)$, one can easily derive the identity

$$\int_b^\infty b_{22}(y)\,B_{22}(y,0)\,\mathrm{d}y = \int_{-h}^0 \frac{\partial B_{22}^{(\mathrm{p})}}{\partial y}(b,z)\,B_{22}(b,z)\,\mathrm{d}z + \int_b^\infty b_{22}(y)\,B_{22}^{(\mathrm{p})}(y,0)\,\mathrm{d}y,$$

that leads to

$$I_6(\Omega) = 2\int_0^\infty b_{22}(y)\,B_{22}(y,0)\,\mathrm{d}y = 2\left[V_1(B_{22}) + \int_b^\infty B_{22}^{(\mathrm{p})}(y,0)\,b_{22}(y)\,\mathrm{d}y\right]. \qquad \text{(A 10)}$$

Since $\{b_{22}(y); B_{22}^{(\mathrm{p})}(y,z)\}$ are known, only $V_1(B_{22})$ must be determined and in order to do so it is desirable to change slightly the weak equation (A 8). If functions with a suffix R are assumed to satisfy the essential condition $\bar{L}_0(\Psi_{\mathrm{R}}) = 0$ one can write

$$\left.\begin{aligned} B_{22}(y,z) &= B_{\mathrm{R}}(y,z)+\bar{B}_0\bar{f}_0(z), \\ \Psi(y,z) &= \Psi_{\mathrm{R}}(y,z)+\bar{L}_0(\Psi)\bar{f}_0(z), \end{aligned}\right\} \tag{A 11}$$

since $\bar{B}_0 = \bar{L}_0(B_{22})$; see (A 3). Placing (A 11) into (A 8) and defining $\{B_{\mathrm{R},j}(y,z); j = 1, 2\}$ as the solutions of the weak equations

$$\left.\begin{aligned} G(B_{\mathrm{R},y}; \Psi_{\mathrm{R}}) &= V_j(\Psi_{\mathrm{R}}); \quad j = 1,2, \\ V_2(\Psi_{\mathrm{r}}) &= -G(\bar{f}_0; \Psi_{\mathrm{R}}), \end{aligned}\right\} \tag{A 12}$$

the following expressions can be obtained:

$$\left.\begin{aligned} B_{22}(y,z) &= B_{\mathrm{R},1}(y,z)+\bar{B}_0[\bar{f}_0(z)+B_{\mathrm{R},2}(y,z)], \\ \bar{B}_0 &= \frac{V_1(\bar{f}_0)+G(B_{\mathrm{R},1}; B_{\mathrm{R},2})}{[G(\bar{f}_0;\bar{f}_0)-G(B_{\mathrm{R},2}; B_{\mathrm{R},2})]-\mathrm{i}D_0}. \end{aligned}\right\} \tag{A 13}$$

From (A 12) and (A 13) it follows that

$$V_1(B_{22}) = G(B_{\mathrm{R},1}; B_{\mathrm{R},1})+\bar{B}_0[V_1(\bar{f}_0)+G(B_{\mathrm{R},1}; B_{\mathrm{R},2})], \tag{A 14}$$

and so $V_1(B_{22})$ can be determined once the coefficients $G_{ij} = G(B_{\mathrm{R},i}; B_{\mathrm{R},j})$ are computed. If $B_{\mathrm{R},j}(y,z), j = 1,2$, belong to the space of functions $W_{\mathrm{R}}(A)$, let $A_{ij}(.;.)$ be the functionals

$$A_{ij}(\phi_{\mathrm{R}}; \Psi_{\mathrm{R}}) = \frac{V_j(\phi_{\mathrm{R}})\, V_i(\Psi_{\mathrm{R}})}{G(\phi_{\mathrm{R}}; \Psi_{\mathrm{R}})}, \quad i,j = 1,2, \tag{A 15}$$

defined in the Cartesian product space $W_{\mathrm{R}}(A)\times W_{\mathrm{R}}(A)$. It is an easy task to check now that $G_{ij} = A_{ij}(B_{\mathrm{R},i}; B_{\mathrm{R},j})$ and, furthermore, that $A_{ij}(.;.)$ is stationary at $(B_{\mathrm{R},i}; B_{\mathrm{R},j})\in W_{\mathrm{R}}(A)\times W_{\mathrm{R}}(A)$. Then an error of order $O(\delta B_{\mathrm{R},i})$ in $B_{\mathrm{R},i}$ implies an error of $O(\delta B_{\mathrm{R},i}\,\delta B_{\mathrm{R},j}) \ll O(\delta B_{\mathrm{R},i}; \delta B_{\mathrm{R},j})$ in G_{ij}. Or, in short, a relatively rough approximation for $B_{22}(y,z)$ implies a much better approximation for I_6. Results shown in Aranha & Pesce (1988) confirm this feature of the present variational method.

The trapped-mode approximation (5.5) is asymptotically correct in the limit $S\rightarrow\infty$, where S is the submergence depth. Placing (5.5) into (3.3) one can check that $b_{22}(y) = 0$ for $y > b$ and so $B_{22}^{(\mathrm{p})}(y,z) \equiv 0$. If S is relatively large it is not difficult to obtain an expression for the proper trial function $\Gamma_{\mathrm{R}}(y,z)$. Indeed, when $S\rightarrow\infty$ the solutions of (A 12) are, respectively,

$$B_{\mathrm{R},1}(y,z) = 0 \text{ (since } b_{22}(y)\rightarrow 0)$$

and

$$\cos D_0 b B_{\mathrm{R},2}(y,z) = (\cos D_0 y-\cos D_b b)\bar{f}_0(z).$$

In this way the choice

$$\Gamma_{\mathrm{R}}(y,z) = (\cos D_0 y-\cos_0 D_0 b)\bar{f}_0(z)$$

seems to be a convenient one for a relatively large S and then the coefficients G_{ij} can be determined from the simple expressions

$$G(B_{\mathrm{R},i}; B_{\mathrm{R},j}) = V_i(\Gamma_{\mathrm{R}})\, V_j(\Gamma_{\mathrm{R}})/G(\Gamma_{\mathrm{R}}; \Gamma_{\mathrm{R}});$$

see (A 15). For a rectangular box in deep water with beam $2b = 1$, width D and distant S from the free surface the following formulas can be obtained for the relevant parameters:

$$\left.\begin{aligned} J &= \mathrm{e}^{-8K_0 S}(1-\mathrm{e}^{-8K_0 D}); \quad K_0 = \Omega^2, \\ V_1(\Gamma_\mathrm{R}) &= -3\sqrt{8}\lambda_0^2\left(\frac{K_0}{D_0}\right)[\sin D_0 b - D_0 b \cos D_0 b], \\ V_2(\Gamma_\mathrm{R}) &= D_0\left(1+J\left(1+\frac{8K_\mathrm{T}^2}{D_0^2}\right)\right)[\sin D_0 b - D_0 b \cos D_0 b], \\ G(\Gamma_\mathrm{R};\Gamma_\mathrm{R}) &= -D_0\left[JD_0 b\left(1+\frac{4K_\mathrm{T}^2}{D_0^2}\right) + D_0 b\cos^2 D_0 b\left(1+J\left(1+\frac{8K_\mathrm{T}^2}{D_0^2}\right)\right)\right. \\ &\qquad \left. -\tfrac{1}{2}\sin 2D_0 b\left(1+2J\left(1+\frac{6K_\mathrm{T}^2}{D_0}\right)\right)\right]. \end{aligned}\right\} \quad (\mathrm{A}\ 16)$$

From (A 10), (A 13), (A 14) it is not difficult to check that I_6 remains bounded even when $G(\Gamma_\mathrm{R};\Gamma_\mathrm{R}) \to 0$. This result is closely related to the existence and uniqueness theorem for the linear diffraction problem. Obviously the approximation for I_6 can be improved but the present one, given by (A 16), seems consistent with the related approximation for the trapped mode. If S is relatively large the parameter J can be disregarded compared to 1, and the simple expression below can be derived:

$$\frac{1}{4I_0(\Omega)}\int_{-\infty}^{\infty} B_{22}(y,0)\, b_{22}(y)\,\mathrm{d}y \approx \sqrt{3}\left(\frac{\lambda_0}{K_0}\right)^5 \Omega^7[a_1 + \mathrm{i}a_2], \quad (\mathrm{A}\ 17a)$$

with

$$\left.\begin{aligned} a_1 &= \frac{1}{1+2\lambda_0 b}\left(1-\frac{1-\cos D_0 b - 4D_0 b\sin^2 D_0 b + (D_0 b)^2 \sin^2 D_0 b}{(2D_0 b\cos D_0 b - \sin D_0 b)^2 + \cos^2 D_0 b}\right), \\ a_2 &= \frac{1}{1+2\lambda_0 b}\,\frac{\sin^2 D_0 b}{(2D_0 b\cos D_0 b - \sin D_0 b)^2 + \cos^2 D_0 b}. \end{aligned}\right\} \quad (\mathrm{A}\ 17b)$$

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